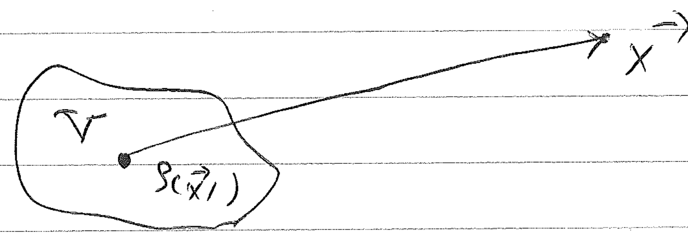
Lecture 7 02/07/2018

Review of Electrostatics (Cont'd)

Multipole Expansion of the Potential

The electric potential due to a localized charge distribution may be decomposed into individual contributions from elementary sources, which is called the <u>multipole expansion</u>. The higher order multipoles contribute more weakly as the distance from the distribution increases. Therefore, only the lowest-order non-vanishing multipole is important at very large distances.



The potential due to the localized distribution within volume V follows:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d\tau'$$

At distances larger than the size of the distribution we have:

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l,m} \frac{r'^{l}}{r^{l+1}} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) \, Y^*_{lm}(\theta',\phi')$$

Thus:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} q_{lm}$$

Here q_{lm} are the lm-multipole moments of $\rho(\vec{x})$:

$$q_{lm} = \int_V \rho(\vec{x}')\, r'^{l}\, Y^*_{lm}(\theta',\phi')\, d\tau'$$

Let us consider some specific low-order multipoles:

$$q_{00} = \int_V \rho(\vec{x}')\, Y^*_{00}(\theta',\phi')\, d\tau' = \frac{Q}{\sqrt{4\pi}} \qquad (Q: \text{total charge in } V)$$

$$q_{10} = \int_V \rho(\vec{x}')\, r'\, Y^*_{10}(\theta',\phi')\, d\tau' = \sqrt{\frac{3}{4\pi}} \int \rho(\vec{x}')\, \underbrace{r'\cos\theta'}_{z'}\, d\tau' = \sqrt{\frac{3}{4\pi}}\, p_z$$

$$q_{1,1} = \int \rho(\vec{x}')\, r'\, Y^*_{11}(\theta',\phi')\, d\tau' = -\sqrt{\frac{3}{8\pi}} \int \rho(\vec{x}')\, \underbrace{r'\sin\theta'\, e^{-i\phi'}}_{x'-iy'}\, d\tau' =$$

$$= -\sqrt{\frac{3}{8\pi}}\,(p_x - i p_y) = -q^*_{1,-1}$$

Here, $\vec{p} = (p_x, p_y, p_z)$, where $\vec{p} = \int_V \rho(\vec{x}')\, \vec{x}'\, d\tau'$ is the total electric dipole in V.

$$q_{20} = \int_V \rho(\vec{x}')\, r'^2\, Y^*_{20}(\theta',\phi')\, d\tau' = \sqrt{\frac{5}{4\pi}} \int_V \rho(\vec{x}')\, \frac{r'^2}{2}\,(3\cos\theta'^2 - 1)\, d\tau'$$

$$= \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int_V \rho(\vec{x}')\,(3z'^2 - r'^2)\, d\tau' = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33}$$

Also, $q_{2,2}$, $q_{2,1}$, $q_{2,-1}$, $q_{2,-2}$ can be written as linear combinations of Q_{ij}, where:

$$Q_{ij} = \int_V \rho(\vec{x}') \left(3x_i' x_j' - \delta_{ij} r'^2 \right) d\tau'$$

The Q_{ij}'s are known as the quadrupole moment tensor of the distribution, which is a 3×3 symmetric and traceless matrix.

The usefulness of $q_{\ell m}$'s is that any coordinate rotation mixes those with the same ℓ amongst each other. This is the property of "irreducibility" for each ℓ.

We note that in general the multipole moments $q_{\ell m}$ depend on the choice of the origin. However, the lowest-order non-vanishing multipole(s) do not depend on the choice of origin. This is easily seen in the case of total charge as q_{00} has no dependence on the origin. Also, if $Q=0$, we have:

$$\vec{P} = \int_V \rho(\vec{x}') \vec{x}' \, d\tau' \xrightarrow[\vec{x}' \to \vec{x}' + \vec{x}_0]{} \int_V \rho(\vec{x}' + \vec{x}_0) (\vec{x}' + \vec{x}_0) \, d\tau' =$$

$$\int_V \rho(\vec{x}') \vec{x}' d\tau' + \vec{x}_0 \underbrace{\int_V \rho(\vec{x}') d\tau'}_{Q=0} = \vec{P}$$

The same can be shown for the quadrupole moment if both the total charge and the dipole moment vanish.

An important and useful identity is the following:

$$\int_V \vec{E}(\vec{x}) d\tau = \frac{-\vec{P}}{3\varepsilon_0}$$

This holds for a spherical volume V that contanins all of the charges, and $\vec{P}$ is the dipole moment due to the charge distribution.

The multipole expansion is also useful for expressing the energy and force of a charge distribution due to an external field. Consider an external electric field $\vec{E}_{ext}$ and a charge distribution $\rho(\vec{x})$ within a volume V in the presence of the external field. The energy of the distribution is:

$$W = \int_V \rho(\vec{x}) \Phi_{ext}(\vec{x}) d\tau$$

Taylor expaning Φ_{ext} about the origin (which is arbitrary) gives:

$$\Phi_{ext}(\vec{x}) = \Phi_{ext}(0) + \vec{x} \cdot \vec{\nabla} \Phi_{ext}(0) + \frac{1}{2} x_i x_j \, \partial_i \partial_j \Phi_{ext}(0)$$

Here we use the summation convention for repeated indices. Then:

$$W = \Phi_{ext}(0) \int_{\mathcal{V}} \rho(\vec{x})\, d\tau + \vec{\nabla} \Phi_{ext}(0) \cdot \int_{\mathcal{V}} \rho(\vec{x})\, \vec{x}\, d\tau + \frac{1}{2} \partial_i \partial_j \Phi_{ext}(0) + \dots$$

$$\int_{\mathcal{V}} \rho(\vec{x})\, x_i x_j \, d\tau$$

Note that:

$$\int_{\mathcal{V}} \rho(\vec{x})\, d\tau = Q \quad , \quad \int_{\mathcal{V}} \rho(\vec{x})\, \vec{x}\, d\tau = \vec{p} \, , \quad \vec{\nabla} \Phi_{ext}(0) = -\vec{E}_{ext}(0)$$

Thus:

$$W = Q \Phi_{ext}(0) - \vec{p} \cdot \vec{E}_{ext}(0) - \frac{1}{6} \partial_i E_{ext,j}(0) \int_{\mathcal{V}} \rho(\vec{x}) (3 x_i x_j - \delta_{ij} r^2 + \delta_{ij} r^2)\, d\tau$$

$$+ \dots = Q \Phi_{ext}(0) - \vec{p} \cdot E_{ext}(0) - \frac{1}{6} \partial_i E_{ext,j}(0) \int_{\mathcal{V}} \rho(\vec{x}) (3 x_i x_j - \delta_{ij} r^2)\, d\tau$$

$$+ \frac{1}{6} \partial_i E_{ext,i}(0) \int_{\mathcal{V}} r^2 \, d\tau + \dots$$

But $\int_{\mathcal{V}} \rho(\vec{x}) (3 x_i x_j - \delta_{ij} r^2)\, d\tau = Q_{ij}$. Also:

$\partial_i E_{ext,i}(0) = \vec{\nabla} \cdot \vec{E}_{ext}(0) = 0$ (charges that produce $\vec{E}_{ext}$ not contained in $\mathcal{V}$)

We therefore find:

$$W = Q \Phi(0) - \vec{p} \cdot \vec{E}_{ext}(0) - \frac{1}{6} \partial_i E_{ext,j}(0) Q_{ij} + \dots$$

The force on the charge distribution is found in a similar way to be:

$$\vec{F} = Q\,\vec{E}_{ext}(0) + \vec{\nabla}(\vec{p} \cdot E_{ext}(0)) + \vec{\nabla}\left(\frac{1}{6}\partial_i E_{ext,j}(0)\, Q_{ij}\right) + \ldots$$

We not that each term in $\vec{F}$ is $-\vec{\nabla}$ of the corresponding term in W, which is expected.